UDC 62-50

ON A CLASS OF LINEAR DIFFERENTIAL GAMES WITH IMPULSE CONTROLS

PMM Vol. 38, №4, 1974, pp. 590-598
V. I. UKHOBOTOV
(Moscow)
(Received March 12, 1973)

We examine a class of linear differential games in which the first player can exert impulse controls while the second player has at his disposal controls with geometric constraints. We formulate a game problem and we prove a theorem which answers the problem posed in the class of games being considered. We present examples. The paper's contents abut those in [1-5].

1. Let the equations of motion have the following form:

$$dz = Bz\,dt + v\,dt + N\,d\Phi, \quad z \in R^n, \quad v \in V \subset R^n \tag{1.1}$$

Here R^n is an n-dimensional Euclidean space, B is a constant square matrix of dimension n, N is a constant matrix having n rows and r columns, V is a convex compactum.

Let $t > 0$ and $C[0, t]$ be a Banach space of continuous r-dimensional vector-valued functions $x(\tau)$, defined on $[0, t]$, with norm $\varkappa([0, t], x(\tau)) = \max_{0 \leqslant \tau \leqslant t} \times \|x(\tau)\|$, where $\|x(\tau)\|$ is the norm in an r-dimensional linear normed space R^r. By $W[0, t]$ we denote the space of r-dimensional vector-valued functions $\Phi(\tau)$ of bounded variation on $[0, t]$; the norm in $W[0, t]$, denoted by $\rho([0, t], \Phi(\tau))$, is generated by the norm $\varkappa([0, t], x(\tau))$ as in the space adjoint to $C[0, t]$.

Let there be given $z_0 \in R^n$, $\sigma > 0$, $\Phi(\tau) \in W[0, \sigma]$ and the vector-valued function $v(\tau) \in V$ measurable on $[0, \sigma]$. We assume that under the action of functions $\Phi(\tau)$, $v(\tau)$ the phase point z of system (1.1) displaces from the initial position to the point

$$z(\sigma) = e^{\sigma B} z_0 + \int_0^\sigma e^{(\sigma-\tau)B} v(\tau)\,d\tau + \int_0^\sigma e^{(\sigma-\tau)B} N\,d\Phi(\tau) \tag{1.2}$$

at instant σ, where the last integral is understood in the Riemann-Stieltjes sense. We introduce the variable μ, varying by the following rule:

$$\mu(\sigma) = \mu_0 - \rho([0, \sigma], \Phi(\tau)) \tag{1.3}$$

We write the constraints on the choice of function $\Phi(\tau)$ in the form of inequalities

$$\mu(\sigma) \geqslant 0 \tag{1.4}$$

By I we denote the set $\mu \geqslant 0$ and the direct product of R^n and I by $R^n \times I$. In the game problem to be examined below we use the following rule. The first player chooses a function $\Phi(\tau)$, the second player the function $v(\tau) \in V$. Let $[z_0;\ \mu_0] \in R^n \times I$ be the initial position of the game. According to his own judgement the second player selects $\sigma_1 > 0$ and the control $v_1(\tau) \in V$ measurable on $[0, \sigma_1]$. He communicates his own choice to the first player. Knowing the second player's choice, the first player chooses the control $\Phi_1(\tau) \in W[0, \sigma_1]$ on $[0, \sigma_1)$ so as to fulfil (1.4). Under the action of the controls chosen the point $[z_0;\ \mu_0] \in R^n \times I$ displaces to the point

$[z(\sigma_1);\ \mu(\sigma_1)] \in R^n \times I$ (see (1.2) and (1.3)). Next, the second player selects $\sigma_2 > 0$ and the control $v_2(\tau)$ on $[0, \sigma_2]$ and informs the first player, etc. These are the so-called σ-strategies.

Let π be the linear mapping of R^n into R^q, where R^q is a q-dimensional linear Euclidean space. For each $t > 0$ we consider the set

$$A(t) = \left\{y \in R^q:\ y = \int_0^t \pi e^{(t-\tau)B} N\, d\Phi(\tau),\ \rho([0, t]\,\Phi(\tau)) = 1\right\} \tag{1.5}$$

Using the weak compactness of a ball in $W[0, t]$, we can show that for each $t > 0$ the set $A(t)$ is a convex compactum in R^q given by the system of inequalities (the asterisk denotes transposition)

$$(y, \psi) \leqslant \max_{0 \leqslant \tau \leqslant t} \| N^* e^{\tau B^*} \pi^* \psi \| \tag{1.6}$$

We set $A(0) = \bigcap_{t>0} A(t)$. Let a closed set G be given in R^q.

Definition 1. A game starting from a point $[z; \mu] \in R^n \times I$ can be completed at an instant $t_1 > 0$ if for any σ-strategy of the second player there exists a σ-strategy of the first player such that $\pi z(t_1) \in G + \mu(t_1) A_1(0)$.

The following problem can be formulated in relation with the definition given.

Problem 1. Given $G \subset R^q$ and $t_1 > 0$; determine the set of those points $[z; \mu] \in R^n \times I$ from which the game can be completed at the instant t_1.

2. We shall solve Problem 1 under the following assumptions:

1) $\pi e^{\tau B} V = y(\tau) + k(\tau) S, \quad \tau \geqslant 0$

2) $\| N^* e^{\tau B^*} \pi^* \psi \| = \beta(\tau)\, c(\psi), \quad \tau \geqslant 0,\ \psi \in R^q$

3) $m(t) = \max_{0 \leqslant \tau \leqslant t} \beta(\tau) > 0, \quad t > 0$

4) $G = a + \varepsilon S, \quad a \in R^q, \quad a = \text{const}$

Here S is some compactum in R^q, convex and symmetric with respect to the origin, containing the zero vector as an interior point, $c(\psi)$ is the support function of S

$$c(\psi) = \max_{s \in S} (s, \psi) \quad \text{for} \quad \psi \in R^q$$

$k(\tau)$ and $\beta(\tau)$ are continuous scalar functions, $k(\tau) \geqslant 0$, $\beta(\tau) \geqslant 0$ for $\tau \geqslant 0$, $y(\tau)$ is a continuous q-dimensional vector-valued function.

From assumptions (1) and (2) and from (1.5), (1.6) we can get that

$$\left\{w \in R^q:\ w = \int_{\tau_1}^{\tau_1+\tau_2} \pi e^{\tau B} v(\tau)\, d\tau,\ v(\tau) \in V\right\} = \int_{\tau_1}^{\tau_1+\tau_2} y(\tau)\, d\tau + \int_{\tau_1}^{\tau_1+\tau_2} k(\tau)\, d\tau \cdot S \tag{2.1}$$

$$\pi e^{\tau_1 B} A(\tau_2) = \max_{\tau_1 \leqslant \tau \leqslant \tau_1+\tau_2} \beta(\tau) \cdot S \tag{2.2}$$

for any $\tau_1 \geqslant 0$, $\tau_2 \geqslant 0$.

By t_2 we denote the largest of the numbers $t \geqslant 0$ for which

$$\varepsilon - \int_0^t k(\tau)\,d\tau \geqslant 0$$

If this inequality is fulfilled for all $t \geqslant 0$, we set $t_2 = +\infty$.

Lemma 1. Let $0 \leqslant t_1 \leqslant t_2$; in order that the game can be completed at the instant t_1 from the point $[\mathbf{z};\ \mu] \in R^n \times \mathbf{I}$, it is necessary and sufficient that

$$\pi e^{t_1 B}\mathbf{z} + \int_0^{t_1} \mathbf{y}(\tau)\,d\tau - \mathbf{a} \in (\mu m(t_1) + \varepsilon - K_1) S, \quad K_1 = \int_0^{t_1} k(\tau)\,d\tau \qquad (2.3)$$

Proof. Using the definition of the operation $\overset{*}{-}$ (see [1]), for example), we can show that

$$(\mu m(t_1) + \varepsilon - K_1) S = (\mu m(t_1) + \varepsilon) S \overset{*}{-} K_1 S \qquad (2.4)$$

$$(\mu m(t_1) + \varepsilon - K_1) S = \mu m(t_1) S + (\varepsilon S \overset{*}{-} K_1 S) \qquad (2.5)$$

Necessity. Suppose that (2.3) is not fulfilled; then from (2.1) and (2.4) follows the existence of the control $\mathbf{v}_1(\tau) \in V$ measurable on $[0, t_1]$, such that

$$\pi e^{t_1 B}\mathbf{z} + \int_0^{t_1} \pi e^{(t_1-\tau)B}\,\mathbf{v}_1(\tau)\,d\tau - \mathbf{a} \,\overline{\in}\, (\mu m(t_1) + \varepsilon) S \qquad (2.6)$$

Suppose the first player chose the control $\Phi_1(\tau) \in W[0, t_1]$ satisfying (1.4). Then there exists $\lambda \in [0, 1]$ such that

$$\lambda\mu = \rho([0, t_1], \Phi_1(\tau)), \qquad \mu(t_1) = (1 - \lambda)\mu$$

From this and from relations (1.5) and (2.2) follows the existence of a vector $s_1 \in S$ such that

$$\int_0^{t_1} \pi e^{(t_1-\tau)B}\, N d\Phi(\tau) = \lambda\mu m(t_1)\, s_1$$

Then from (2.6) it follows that

$$\pi\mathbf{z}(t_1) = \pi e^{t_1 B}\mathbf{z} + \int_0^{t_1} \pi e^{(t_1-\tau)B}\,\mathbf{v}_1(\tau)\,d\tau + \lambda\mu m(t_1)\, s_1 \,\overline{\in}\, \mathbf{a} + (1-\lambda)\mu A(0) + \varepsilon S$$

because for any $\lambda \in [0, 1]$ and $s \in S$ we have

$$\mathbf{a} - \lambda\mu m(t_1)\, s + (1-\lambda)\mu A_1(0) + \varepsilon S \subset \mathbf{a} + (\mu m(t_1) + \varepsilon) S$$

Sufficiency. Suppose (2.3) holds; then from (2.5) follows the existence of $s_1 \in S$ such that

$$\pi e^{t_1 B}\mathbf{z} + \int_0^{t_1} \pi e^{(t_1-\tau)B}\,\mathbf{v}(\tau)\,d\tau + \mu m(t_1)\, s_1 \in \mathbf{a} + \varepsilon S \qquad (2.7)$$

for all functions $\mathbf{v}(\tau) \in V$ measurable on $[0, t_1]$. The proof of the sufficiency follows from (2.7) if we make use of (1.5) and (2.2).

Let us now consider the case $t_1 > t_2$.

Theorem 1. In order that the game from the point $[\mathbf{z};\ \mu] \in R^n \times \mathbf{I}$ can be completed at an instant $t_1 > t_2$, it is necessary and sufficient that

$$\pi e^{t_1 B}\mathbf{z} + \int_0^{t_1} \mathbf{y}(\tau)\,d\tau - \mathbf{a} + m(t_1)\int_{t_2}^{t_1} \frac{k(\tau)}{m(\tau)}\,d\tau \cdot S \subset \mu m(t_1) S \qquad (2.8)$$

Proof. Necessity. For each integer $j \geqslant 1$ we define

$$b(j)=\sum_{i=1}^{j}\Big[\Big(\int_{t_2+(i-1)\sigma}^{t_2+i\sigma}k(\tau)\,d\tau\Big)\Big/m(t_2+i\sigma)\Big],\qquad \sigma=\frac{t_1-t_2}{j}$$

As is easy to see

$$\lim_{j\to\infty}b(j)=\int_{t_2}^{t_1}\frac{k(\tau)}{m(\tau)}\,d\tau$$

Suppose that (2.8) is not fulfilled; then a number j_1 exists such that

$$\pi e^{t_1B}\mathbf{z}+\int_0^{t_1}\mathbf{y}(\tau)\,d\tau-\mathbf{a}+m(t_1)b(j_1)S\equiv\mu m(t_1)S \tag{2.9}$$

We set

$$\eta_i=\Big(\int_{t_2+(i-1)\sigma_1}^{t_2+i\sigma_1}k(\tau)\,d\tau\Big)\Big/m(t_2+i\sigma_1),\qquad \sigma_1=\frac{t_1-t_2}{j_1}$$

Then we can write $b(j_1)=\eta_1+\eta_2+\ldots+\eta_{j_1}$.

From (2.9) and the convexity of compactum S follows the existence of a nonzero vector $\psi_1\in R^q$ and a number $\nu>0$ such that

$$\Big(\pi e^{t_1B}\mathbf{z}+\int_0^{t_1}\mathbf{y}(\tau)\,d\tau-\mathbf{a},\ \psi\Big)+m(t_1)b(j_1)c(\psi_1)\geqslant\mu m(t_1)c(\psi_1)+\nu \tag{2.10}$$

Let the vector $s_1\in S$ be such that $c(\psi_1)=(s_1,\psi_1)$. From (2.1) follows the existence of a function $\mathbf{v}_1(\tau)$ measurable on $[0,\sigma_1]$, such that

$$\pi e^{(t_1-\sigma_1)B}\int_0^{\sigma_1}e^{(\sigma_1-\tau)B}\mathbf{v}_1(\tau)\,d\tau=\int_{t_1-\sigma_1}^{t_1}\mathbf{y}(\tau)\,d\tau+\int_{t_1-\sigma_1}^{t_1}k(\tau)\,d\tau\cdot s_1 \tag{2.11}$$

Let us show that if the second player takes σ_1 and $\mathbf{v}_1(\tau)\in V$ on $[0,\sigma_1]$, then for any control

$$\pi e^{(t_1-\sigma_1)B}\mathbf{z}(\sigma_1)+\int_0^{t_1-\sigma_1}\mathbf{y}(\tau)\,d\tau-\mathbf{a}+m(t_1-\sigma_1)(b(j_1)-\eta_{j_1})S\equiv\mu(\sigma_1)m(t_1-\sigma_1)S \tag{2.12}$$

By virtue of (1.4) and (2.2) we can assume that there exist $s\in S$ and $\lambda\in[0,1]$ such that

$$\mu(\sigma_1)=(1-\lambda)\mu \tag{2.13}$$

$$\pi e^{(t_1-\sigma_1)B}\int_0^{\sigma_1}e^{(\sigma_1-\tau)B}N\,d\Phi(\tau)=\lambda\mu\max\nolimits_{t_1-\sigma_1\leqslant\tau\leqslant t_1}\beta(\tau)\cdot s \tag{2.14}$$

Then, from (2.11) and (2.14) follows

$$\pi e^{(t_1-\sigma_1)B}\mathbf{z}(\sigma_1)+\int_0^{t_1-\sigma_1}\mathbf{y}(\tau)\,d\tau=\pi e^{t_1\mathbf{B}}\mathbf{z}+\int_0^{t_1}\mathbf{y}(\tau)\,d\tau+ \tag{2.15}$$

$$\int_{t_1-\sigma_1}^{t_1}k(\tau)\,d\tau\cdot s_1+\lambda\mu\max\nolimits_{t_1-\sigma_1\leqslant\tau\leqslant t_1}\beta(\tau)\cdot s$$

Let $\mu\leqslant b(j_1)-\eta_{j_1}$; then (2.12) always holds since

$$\mu(\sigma_1)=(1-\lambda)\mu\leqslant b(j_1)-\eta_{j_1}$$

We now examine the case when $\mu\geqslant b(j_1)-\eta_{j_1}$. If $\lambda\in[0,1]$ is such that $\mu(\sigma_1)<b(j_1)-\eta_{j_1}$, then (2.12) holds. Let us consider $\lambda\in[0,1]$ such that

$\mu(\sigma_1) \geqslant b(j_1) - \eta_{j_1}$, i.e.

$$0 \leqslant \lambda\mu \leqslant \mu - (b(j_1) - \eta_{j_1}) \tag{2.16}$$

from (2.15) and (2.10) it follows that

$$\begin{aligned}
&(\pi e^{(t_1-\sigma_1)B}\mathbf{z}(\sigma_1), \psi_1) + \Big(\int_0^{t_1-\sigma_1} \mathbf{y}(\tau)\,d\tau - \mathbf{a}, \psi_1\Big) + \\
&m(t_1-\sigma_1)(b(j_1)-\eta_{j_1})c(\psi_1) - \\
&\mu(\sigma_1)m(t_1-\sigma_1)c(\psi_1) \geqslant \mu(t_1)m(t_1)c(\psi_1) - m(t_1)b(j_1)c(\psi_1) + \\
&\int_{t_1-\sigma_1}^{t_1} k(\tau)\,d\tau c(\psi_1) - \lambda\mu \max_{t_1-\sigma_1\leqslant\tau\leqslant t_1}\beta(\tau)c(\psi_1) + \\
&m(t_1-\sigma_1)(b(j_1)-\eta_{j_1})c(\psi_1) - \\
&(1-\lambda)\mu m(t_1-\sigma_1)c(\psi_1) + \nu = g(\lambda)
\end{aligned}$$

We can show that the inequality $g(\lambda) \geqslant \nu$ is fulfilled for $\lambda \in [0, 1]$ and satisfying (2.16); this signifies that (2.12) holds.

Repeating this argument j_1 times we find the second player's σ-strategy such that

$$\pi e^{t_2B}\mathbf{z}(t_1-t_2) + \int_0^{t_2} \mathbf{y}(\tau)\,d\tau - \mathbf{a} \in \mu(t_1-t_2)m(t_2)S$$

is fulfilled at the instant $t_1 - t_2$ for any σ-strategy of the first player; here $\sigma_i = \sigma_1$. The application of Lemma 1 completes the proof of necessity.

S u f f i c i e n c y. Suppose (2.8) holds; then

$$\mu \geqslant \int_{t_2}^{t_1} \frac{k(\tau)}{m(\tau)}\,d\tau \tag{2.17}$$

must necessarily be fulfilled. Since $m(t_1) \geqslant m(\tau)$ for $0 \leqslant \tau \leqslant t_1$,

$$\int_{t_1-\sigma}^{t_1} k(\tau)\,d\tau + m(t_1)\int_{t_2}^{t_1-\sigma} \frac{k(\tau)}{m(\tau)}\,d\tau \leqslant m(t_1)\int_{t_2}^{t_1}\frac{k(\tau)}{m(\tau)}\,d\tau \tag{2.18}$$

for any $\sigma \in [t_2, t_1]$. From (2.8), (2.17) and (2.18) follows

$$\begin{aligned}
&\pi e^{t_1B}\mathbf{z} + \int_{t_2}^{t_1}\mathbf{y}(\tau)\,d\tau - \mathbf{a} + \int_{t_2}^{t_1} k(\tau)\,d\tau \times \\
&\times S \subset \Big(\mu m(t_1) - m(t_1)\int_{t_2}^{t_1-\sigma}\frac{k(\tau)}{m(\tau)}\,d\tau\Big)S
\end{aligned} \tag{2.19}$$

Suppose that the second player had selected $0 < \sigma_1 \leqslant t_1 - t_2$ and the control $\mathbf{v}_1(\tau) \in V$ on $[0, \sigma_1]$; then there exists $s_1 \in S$ such that (2.11) holds.

Let us first consider the case $m(t_1-\sigma_1) < m(t_1)$. Then, according to (2.19), there exists $s_2 \in S$ such that

$$\pi e^{t_1B}\mathbf{z} + \int_{t_2}^{t_1}\mathbf{y}(\tau)\,d\tau - \mathbf{a} + \int_{t_1-\sigma_1}^{t_1} k(\tau)\,d\tau s_1 + m(t_1)\Big(\mu - \int_{t_2}^{t_1-\sigma_1}\frac{k(\tau)}{m(\tau)}\,d\tau\Big)s_2 = \theta \tag{2.20}$$

where $\theta \in R^q$ is the zero vector. For $s_2 \in S$ we can find a function $\Phi(\tau) \in W[0, \sigma_1]$, $\rho([0, \sigma_1], \Phi(\tau)) = 1$ such that

$$\pi e^{(t_1-\sigma_1)B} \int_0^{\sigma_1} e^{(\sigma_1-\tau)B} N \, d\Phi(\tau) = \max_{t_1-\sigma_1 \leqslant \tau \leqslant t_1} \beta(\tau) \cdot s_2 = m(t_1) s_2 \tag{2.21}$$

To the first player we assign the control

$$\Phi_1(\tau) = \left(\mu - \int_{t_2}^{t_1-\sigma_1} \frac{k(\tau)}{m(\tau)} d\tau\right) \Phi(\tau)$$

Then, by virtue of (2.20) and (2.21)

$$\pi e^{(t_1-\sigma_1)B} \mathbf{z}(\sigma_1) + \int_0^{t_1-\sigma_1} \mathbf{y}(\tau) d\tau - \mathbf{a} = \theta$$

In addition

$$\mu(\sigma_1) = \int_{t_1}^{t_1-\sigma_1} \frac{k(\tau)}{m(\tau)} d\tau$$

Consequently,

$$\pi e^{(t_1-\sigma_1)B} \mathbf{z}(\sigma_1) + \int_0^{t_1-\sigma_1} \mathbf{y}(\tau) d\tau - \mathbf{a} + m(t_1-\sigma_1) \int_{t_1}^{t_1-\sigma_1} \frac{k(\tau)}{m(\tau)} d\tau \cdot S \subset \mu(\sigma_1) m(t_1-\sigma_1) S \tag{2.22}$$

Now let $m(t_1-\sigma_1) = m(t_1)$; then we assign $\Phi_1(\tau) \equiv 0$ to the first player. Since strict equality obtains in (2.18) when $m(t_1-\sigma_1) = m(t_1)$, the left hand side of inclusion (2.22) equals

$$\pi e^{t_1 B} \mathbf{z} + \int_0^{t_1} \mathbf{y}(\tau) d\tau - \mathbf{a} + \int_{t_1-\sigma_1}^{t_1} k(\tau) d\tau \cdot s_1 + m(t_1) \int_{t_2}^{t_1-\sigma_1} \frac{k(\tau)}{m(\tau)} d\tau \cdot S$$

Thus we have shown that the first player can always maintain the inclusion (2.22). Consequently, the inclusion

$$\pi e^{t_2 B} \mathbf{z}(t_1-t_2) + \int_0^{t_2} \mathbf{y}(\tau) d\tau - \mathbf{a} \in \mu(t_1-t_2) m(t_2) S$$

is fulfilled at the instant $t_1 - t_2$. The proof is completed by an application of Lemma 1.

We consider now the case $\varepsilon = 0$.

Theorem 2. (1) Let

$$\lim_{\delta \to 0} \int_\delta^{t_1} \frac{k(\tau)}{m(\tau)} d\tau < +\infty$$

In order that the game from the point $[\mathbf{z}; \mu] \in R^n \times \mathrm{I}$ can be completed at an instant t_1, it is necessary and sufficient that

$$\pi e^{t_1 B} \mathbf{z} + \int_0^{t_1} \mathbf{y}(\tau) d\tau - \mathbf{a} + m(t_1) \int_0^{t_1} \frac{k(\tau)}{m(\tau)} d\tau \cdot S \subset \mu m(t_1) S$$

2) Let

$$\lim_{\delta \to 0} \int_\delta^{t_1} \frac{k(\tau)}{m(\tau)} d\tau = +\infty$$

Then it is impossible to complete the game at an instant $t_1 > 0$ from any point $[\mathbf{z}; \mu] \in R^n \times \mathrm{I}$.

The proof of this theorem is analogous to that of Theorem 1.

3. We present several examples illustrating Theorems 1 and 2.

Example 1. Let the equations of motion be of form (1.1). We assume V to be a convex compactum in R^n, symmetric with respect to the origin. Let $R^q = R^1$ and let set G be the segment $[\varepsilon_1, \varepsilon_2]$. For $q = 1$ the matrix π is an n-dimensional row-vector. We denote the segment $[-1, 1]$ by S ; then $G = a + \varepsilon S$, where $a = (\varepsilon_1 + \varepsilon_2) / 2$, $\varepsilon = (\varepsilon_2 - \varepsilon_1) / 2$. It is not difficult to verify that the assumptions (1), (2), (4), stated in Sect. 2, are fulfilled, and

$$\beta(\tau) = \| N^* e^{\tau B^*} \pi^* \|, \quad \mathbf{y}(\tau) = 0, \quad k(\tau) = \max_{\mathbf{v} \in V} (e^{\tau B^*} \pi^*, \mathbf{v})$$

For the fulfillment of assumption (3) we require $\beta(\tau) > 0$ for $0 < \tau \leqslant \sigma$, where σ is some number.

Example 2. Consider a game in which the equations of motion are

$$d\mathbf{z}_1 = \mathbf{z}_2 dt + \mathbf{v} dt, \quad d\mathbf{z}_2 = k_1 \mathbf{z}_1 dt + k_2 \mathbf{z}_2 dt + d\Phi$$

where $\mathbf{z}_1$, $\mathbf{z}_2$ are r-dimensional vectors, k_1 and k_2 are some numbers. It is easy to see that

$$N = \left\| \begin{matrix} \Theta \\ E \end{matrix} \right\|$$

where E and Θ are the r-dimensional unit and zero matrices, respectively. In R^r we define the set

$$S = \{\mathbf{w} \in R^r : (\mathbf{w}, \psi) \leqslant \| \psi \| \text{ for } \psi \in R^r\}$$

We assume that $\mathbf{v} \in \delta S$, where $\delta > 0$. We set $q = r$ and $\pi = (E, 0)$; then $\pi \mathbf{z} \in \varepsilon S$ signifies that $\mathbf{z}_1 \in \varepsilon S$. Assumptions (1) – (4) are fulfilled in this example, and

$$\pi e^{\tau B} V = \delta \, | \alpha(\tau) | \, S, \quad \| N^* e^{\tau B^*} \pi^* \psi \| = | \gamma(\tau) | \, \| \psi \|$$

Here $\alpha(\tau)$ and $\gamma(\tau)$ are the solutions of the following equations:

$$\alpha^{\cdot\cdot} = k_1 \alpha + k_2 \alpha^{\cdot}, \quad \alpha(0) = 1, \quad \alpha^{\cdot}(0) = 0$$
$$\gamma^{\cdot\cdot} = k_1 \gamma + k_2 \gamma^{\cdot}, \quad \gamma(0) = 0, \quad \gamma^{\cdot}(0) = 1$$

Example 3. Let the equations of motion have the form

$$d\mathbf{z}_1 = \mathbf{z}_2 dt, \quad d\mathbf{z}_2 = -k_1 \mathbf{z}_2 dt + B_1 \mathbf{z}_2 dt + d\Phi$$
$$d\mathbf{y}_1 = \mathbf{y}_2 dt, \quad d\mathbf{y}_2 = -k_2 \mathbf{y}_2 dt + B_2 \mathbf{y}_2 dt + \mathbf{v} dt$$

Here $\mathbf{z}_1, \mathbf{z}_2, \mathbf{y}_1, \mathbf{y}_2$ are r-dimensional vectors, k_1, k_2 are some numbers, B_1 and B_2 are constant ($r \times r$)-matrices satisfying the conditions

$$B_i^* = -B_i, \quad B_i^2 = -\omega_i^2 E, \quad i = 1, 2$$

where ω_i are certain nonnegative numbers, E is the r-dimensional unit matrix. We assume that $\mathbf{v} \in \delta S$, where S is the r-dimensional closed Euclidean sphere of unit radius, $\delta > 0$. We take the Euclidean norm as the norm in R^r ; then (see [6], for example)

$$\rho([0, t], \Phi(\tau)) = \int_0^t \left(\sum_{i=1}^r d\Phi_i^2(\tau) \right)^{1/2}$$

We set $q = r$ and $\pi = (E, 0, -E, 0)$; then $\pi \mathbf{z} \in \varepsilon S$ signifies that $\mathbf{z}_1 - \mathbf{y}_1 \in \varepsilon S$. We can verify that

$$\pi e^{\tau B} V = \delta \alpha(\tau) S, \quad \| N^* e^{\tau B^*} \pi^* \psi \| = \beta(\tau) \| \psi \|$$

where

$$\alpha(\tau) = (f_2^2(\tau) + g_2^2(\tau))^{1/2}, \quad \beta(\tau) = (f_1^2(\tau) + g_1^2(\tau))^{1/2}$$

$$f_i(\tau) = \int_0^\tau e^{-k_i t} \cos \omega_i t\, dt, \quad g_i(\tau) = \int_0^\tau e^{-k_i t} \sin \omega_i t\, dt, \quad i = 1, 2$$

REFERENCES

1. Pontriagin, L. S., On linear differential games, 2. Dokl. Akad. Nauk SSSR, Vol. 195, № 4, 1967.
2. Krasovskii, N. N. and Tret'iakov, V. E., On the pursuit problem under constraints on the control impulses. Differentsial'nye Uravneniia, Vol. 2, № 5, 1966.
3. Krasovskii, N. N., Games-theoretical encounter of motions with bounded impulses. PMM Vol. 32, № 2, 1968.
4. Pozharitskii, G. K., On a problem of the impulse contact of motions. PMM Vol. 35, № 5, 1971.
5. Pozharitskii, G. K., Game problem of "soft" impulse contact of two material points. PMM Vol. 36, № 2, 1972.
6. Krasovskii, N. N., Theory of Control of Motion. Moscow, "Nauka", 1968.

Translated by N. H. C.

UDC 62-50

CONTROLLABILITY OF A NONLINEAR SYSTEM IN A LINEAR APPROXIMATION

PMM Vol. 38, № 4, 1974, pp. 599-606
E. L. TONKOV
(Tambov)
(Received November 2, 1973)

We study the conditions for the controllability of a dynamic system whose behavior in a finite-dimensional phase space is described by a nonlinear differential equation. The results obtained complement the investigations in [1−10].

1. Definitions and formulations of results. Let R^n be an n-dimensional arithmetic space of points $x = \operatorname{col}(x_1, \ldots, x_n)$ with norm $|\cdot|$. We examine the system

$$x^{\cdot} = A(t)x + B(t)u + \varphi(t, x, u), \; x \in R^n, \; u \in R^m, \; t \in [t_0, \infty) \quad (1.1)$$

Here the real ($n \times n$) and ($n \times m$) matrices $A(t)$ and $B(t)$ are continuous for $t \in [t_0, \infty)$; the real function $\varphi(t, x, u)$ is continuous in the collection of arguments $(t, x, u) \in [t_0, \infty) \times R^n \times R^m$. We say that the control $u_0(t)$, $t \in I = [t_0, t_1]$ translates the position (t_0, x_0) of system (1.1) into the position $(t_1, 0)$ if the solution $x_0(t)$, satisfying the initial condition $x(t_0) = x_0$ of system (1.1) under control $u = u_0(t)$ is defined for all $t \in I$, is unique on I, and passes through the point $x_1 = 0$ at instant $t_1 : x_0(t_1) = 0$.